

# A Simple Solution of the Arrival Time Problem

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**Abstract:** Based on the principle that arrival time and position are simultaneously measurable quantities a simple formula is derived for the arrival time probability density in nonrelativistic quantum theory.

The observation of time correlations belongs to the most basic type of physical experiments. Imagine a beta-radioactive nucleus  ${}_Z X^A$  whose product nucleus  ${}_{Z+1} Y^A$  decays by alpha-emission. In the most simple experiment one establishes an alpha-detector at a distance  $r_D$  from the nucleus whose click signalizes the arrival of the alpha-particle. An electron detector in the immediate vicinity of the nucleus provides the signal of the *moment of preparation* of the nucleus  $Y$ . The time  $t$  elapsed between the signals of the two detector is called *the arrival time* of the alpha-particle. Notice that this time interval is typically very much longer than the time  $t_D$  which would be required to cover the distance  $r_D$  by the alpha-particle with its mean velocity. One is inclined to think that the alpha-particle appears only at the moment  $(t - t_D)$  of the decay of  ${}_Z X^A$  but in the situation, we are considering, no empirical significance can be attributed to the "moment of decay" itself: It can only be inferred from the value of  $t$  provided the velocity is known. Therefore, the only consistent point of view is to admit that the particle has been on its way to the point at  $r_D$  throughout the whole time interval  $t$ .

In spite of the fundamental significance of observations of this kind in quantum theory we do not have any rule to calculate arrival time probability densities<sup>1</sup>. Assume that the alpha-emission can be described by a single-particle potential model and let us choose the signal of the beta-detector for the zero moment of time. Then at  $t = 0$  the wave-function  $\psi(\mathbf{r}, t)$  of the alpha-particle will be concentrated within a spherical potential wall around the point  $r = 0$  where the nucleus is found. This wave function is the only

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<sup>1</sup>A detailed explication of this problem is found in *G. R. Allcock*, The Time of Arrival in Quantum Mechanics, *Ann. Phys.*, **53**, 253, 286, 311 (1969)

information we have on the alpha-particle. How to deduce from it the time distribution of the clicks of the alpha-detector?

In order to concentrate on the time behaviour alone without complexities of the angular distribution we assume isotropic initial conditions which lead to spherically symmetric wave-function  $\psi(r, t)$ . Accordingly, the pointlike detector will be replaced by a thin spherically symmetrical shell of mean radius  $r_D$ . Let us denote the arrival time probability density<sup>2</sup> by  $\text{prob}(t|r_D, I)$ . The function  $\text{prob}(t|r, I)$  is a conditional probability density of the random variable  $t$  which is normalized as<sup>3</sup>

$$\int_{t_{min}}^{\infty} dt \text{ prob}(t|r, I) = 1.$$

The unspecified condition  $I$  contains the relevant informations on the protocol of the experiment as described above. The coordinate of the detector which is fixed during the time correlation measurement must be classified also among the conditions of the experiment and no normalization in it is required.

The problem is how to calculate the probability density  $\text{prob}(t|r, I)$  from the known  $\psi(r, t)$ . Standard rules of the quantum theory connect  $\psi$  with the probability density

$$\text{prob}(r|t, I) = |\psi(r, t)|^2, \quad (1)$$

of the space distribution at a fixed moment of time rather than with  $\text{prob}(t|r, I)$  and presume the normalization condition

$$\int d^3x \text{ prob}(r|t, I) = \int d^3x |\psi(r, t)|^2 = 1.$$

Reflecting on the radioactive decay (or on the motion of the wave-packets), one is often inclined to identify the arrival time probability density  $\text{prob}(t|r, I)$  at a given  $r$  with the spatial probability density  $\text{prob}(r|t, I)$  (as given is (1)) at a definite moment of time but without justification this step would be

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<sup>2</sup>No notational distinction will be made between time and arrival time since the latter will be referred to the zero moment.

<sup>3</sup>The integral may diverge at  $t_{min} = 0$ . This is a spurious effect due to the instantaneous spreading of wave-packets in nonrelativistic quantum theory and may be avoided if the considerations are confined to the domain  $t > t_{min}$  where  $t_{min}$  is much larger than  $r/c$  ( $c$  is the speed of light) but still much smaller than the lifetime.

grossly in error. From the elements of probability theory it is well known that the probabilities  $\text{prob}(a|b)$  and  $\text{prob}(b|a)$  are as a rule different from each other. Moreover, in quantum theory the time plays the role of the parameter and probabilities are only defined for dynamical variables at definite moments of time. It is just the parametric nature of the time which makes arrival time such an awkward problem.

In spite of all this, physical properties of the absolute square of the wavefunction  $\psi(r, t)$  strongly suggest that  $\text{prob}(t|r, t)$  is indeed proportional to  $|\psi(r, t)|^2$ . At a fixed  $r$  and for times of the order of the lifetime  $\tau$   $|\psi(r, t)|^2$  decreases in time approximately as  $\exp(-t/\tau)$  while its maximum gets farther and farther from the origin<sup>4</sup>. It is not normalized in  $t$  to unity but can be done so. In other words the prescription

$$\text{prob}(t|r, I) = \frac{1}{N(r)} |\psi(r, t)|^2 \quad (2)$$

together with the normalization factor

$$N(r) = \int_{t_{min}}^{\infty} dt |\psi(r, t)|^2 \quad (3)$$

would, from the observational point of view, provide an excellent description of the arrival time probability density.

It seems to me that these formulae can be justified from the theoretical point of view too, accepting the more than plausible principle that *the arrival time and position are simultaneously measurable quantities*. One might think that this principle contradicts the uncertainty relation between position and momentum but it does not. From the experiment we are considering (i.e. from a measurement of the decay law of a radioactive nucleus) no information can be inferred on the momentum of the emitted particle. A more general argument is that, using time of flight spectrometers, we actually measure the position in two subsequent moments of time rather than momentum itself. Substantial additional knowledge (the absence of a force field along the path) is required *to infer* the value of the momentum prior to the detector response.

What the above principle does exclude is that arrival time is an operator in the Hilbert-space which does not commute with the coordinate operators. But this mathematical property of the observables under study could only

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<sup>4</sup>L. Fonda, G. C. Ghirardi and A. Rimini, Decay theory of unstable quantum systems, *Rep. Prog. Phys.*, **41**, 587 (1978).

be maintained if an experiment designed to measure precisely either of them excluded the possibility to measure the other. No such principal incompatibility seems to exist for the simultaneous observation of the arrival time and position.

If arrival time and position are indeed simultaneously measurable entities then their joint probability density  $\text{prob}(r, t|I)$  is a sensible quantity whose existence permits us to introduce the conditional probability densities  $\text{prob}(t|r, I)$  and  $\text{prob}(r|t, I)$  as

$$\text{prob}(t|r, I) = \frac{\text{prob}(r, t|I)}{\text{prob}(r|I)}, \quad (4)$$

$$\text{prob}(r|t, I) = \frac{\text{prob}(r, t|I)}{\text{prob}(t|I)}. \quad (5)$$

On the first line  $\text{prob}(r|I)$  is the probability density to find the alpha-particle (under the conditions  $I$  of the experiment) in  $r$  at some moment of time:

$$\text{prob}(r|I) = \int_{t_{min}}^{\infty} dt \text{ prob}(r, t|I) = \int_{t_{min}}^{\infty} dt \text{ prob}(r|t, I) \cdot \text{prob}(t|I). \quad (6)$$

The function  $\text{prob}(t|I)$  on the second line is equal to the probability density to find the alpha-particle at the moment  $t$  somewhere in space. The only natural possibility for this probability is that it does not depend on time since the particle exists no more in one moment of time than in another<sup>5</sup>. Since a uniform density is not normalizable in the semiinfinite interval  $t_{min} < t < \infty$  we are compelled to confine ourselves to an arbitrarily large but finite time interval  $t_{min} < t < T$ . Then  $\text{prob}(t|I) \approx 1/T$ , but the arbitrary parameter  $T$  drops out of the final formula.

Now, eliminating  $\text{prob}(r, t|I)$  from (4) and (5) we obtain the Bayes-like formula

$$\text{prob}(t|r, I) = \frac{\text{prob}(r|t, I) \cdot \text{prob}(t|I)}{\text{prob}(r|I)}.$$

Inserting (1), (6) and  $\text{prob}(t|I) = 1/T$  into this equation, we arrive at the desired result (2), (3) which seems to be the most natural solution of the arrival time problem.

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<sup>5</sup>For a close analogy remember that in the elementary theory of the radioactive decay the decay probability  $\lambda \cdot dt$  in the interval  $(t, t + dt)$  is independent of  $t$ .

Equations (2) and (3) are applicable also in the case when a high potential barrier is present between the decaying nucleus and the detector so that the particle can reach the detector only through tunneling. Properties of the tunneling time<sup>6</sup> can then be studied by calculating  $\text{prob}(t|r, I)$  both in the presence of the barrier and without it.

The above considerations have to be implemented by the detector efficiency  $\epsilon$ . Following common practice, the theoretical probability distribution (2) must be multiplied by  $\epsilon$  which takes values in the interval  $(0, 1)$ : In  $N$  trials the detector responds in  $\epsilon N$  cases. If it is desirable to protect the wave-function from distortions due to the presence of the detector the efficiency must be close to zero.

In justifying (2) I have been guided by the conviction that this formula seems capable of reproducing the properties of the arrival time known from experience. Its "simplicity" is the natural consequence of this effort but I hope a sufficiently solid foundation has been given to it. On the other hand, my proposal is connected also to the state reduction hypotheses which is probably the only problem within quantum mechanics which has remained controversial since the time of its birth. The recipe (2) does not circumvent this problem: When the detector clicks the wave-function  $\psi(r, t)$  collapses into the domain of the detector and this process is outside the scope of the Schrödinger-equation. Yet this recipe contains an essentially new element since the moment of the collapse is now "chosen by the system itself" rather than by the (hypothetical) intervention of the observer.

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<sup>6</sup>For a review, see *R. Landauer and Th. Martin*, Barrier interaction time in tunneling, *Rev. Mod. Phys.*, **66**, 217 (1994).